

MSFT : MOYAL STAR FORMULATION OF STRING FIELD THEORY ¹

The Moyal star formulation of string field theory is reviewed. The various versions of the star product are compared and related to one another in a regulated theory that resolves associativity anomalies. A summary of computations and challenges is given.

1 Introduction

Witten's formulation of open bosonic string field theory (SFT) [1]-[4] has experienced a rebirth through new physical insights and technical advances [5]-[18]. Its application to certain nonperturbative aspects of string theory, in particular D-branes, has prompted the development of new mathematical tools to reformulate and analyze perturbative as well as nonperturbative string physics in the context of string field theory.

Over a year ago it was shown in [8] that the star product in SFT, originally defined by Witten as a path integral that sews two strings into a third one, is equivalent to the Moyal star product [20] which is at the heart of the familiar formulation of noncommutative geometry [21]. We will refer to this formulation of SFT as the Moyal Star Formulation of String Field Theory (MSFT) [8],[10]-[15],[18].

In [8] it was shown that the Moyal star in MSFT is defined in the phase space of *only even string modes* (x_e^μ, p_e^μ) , $e = 2, 4, 6, \dots$, *for each mode independently*, and furthermore, that the product is local at the midpoint of the string \bar{x}^μ . The midpoint details were fully clarified in [11]. Thus, in MSFT a string field is denoted by $A(\bar{x}, \xi)$, where $\xi_i = (x_2, x_4, x_6, \dots, p_2, p_4, p_6, \dots)$ is the noncommutative space formed by the direct product of Moyal planes. The Moyal star product $(A(\bar{x}) \star B(\bar{x}))(\xi)$ is defined for each pair (x_e^μ, p_e^μ) independently for each e , at fixed \bar{x}^μ .

In the discrete basis labelled by e a cutoff method was introduced to regulate associativity anomalies [10] and provide a precise definition of MSFT with a regulator [10][11]. Subsequently, a Moyal star related to the original one in [8] was discussed in continuous bases [12]-[15]. With proper care of anomalies and regulators, the discrete and continuous bases are completely equivalent as discussed in [11] and in the next section. So, MSFT may be pursued in

different bases and it may be sometimes beneficial to change bases for the efficiency or clarity of a computation. So far explicit computations have been carried out mainly in the original discrete basis which provides a natural and consistent regulator that is necessary for careful computation.

The MSFT star product is similar to, but is different than, deformation quantization [22] of string theory, since in deformation quantization all phase space degrees of freedom enter in the Moyal star, while in MSFT the degrees of freedom involved in the Moyal star are *half* of all phase space of the *excited* modes of the open string (also the midpoint is not part of this space).

The advantage of MSFT is its structural and computational simplicity. Computations in MSFT are based only on the use of the Moyal star product. The new star provides an alternative to the oscillator tool or the conformal field theory tool as a method of computation.

To perform explicit computations, a monoid algebra (almost a group) was introduced in [8][11] as an effective tool. The structure of the monoid algebra turns out to be sufficient to evaluate the star products and traces in most computations of physical interest.

In [11] it was shown that interaction vertices in MSFT are in full agreement with other versions of SFT, in particular the oscillator version in [1]. Furthermore new results involving interaction vertices for any number of perturbative or nonperturbative string states (projectors that describe D-branes) were obtained through MSFT for the first time. The computation of Feynman graphs for interacting strings, with perturbative or nonperturbative external states, has also been developed more recently using MSFT techniques [18]². As an aside it was shown that the propagator in MSFT is also in full agreement with other approaches to string theory. Therefore MSFT, with a cutoff, reproduces the correct vertices and propagators of perturbative string theory while providing a convenient formalism for pursuing nonperturbative aspects.

The progress in MSFT during the past year will be outlined, and some open problems will be mentioned at the end.

²The computation of perturbative Feynman graphs is pursued in the oscillator formulation in [17].

2 Action

The action of open string field theory is given by [1]

$$S = \int (d\bar{x}^\mu) e^{-i\frac{3}{2}\bar{x}_{27}} \text{Tr} \left(\frac{1}{2} A \star (\mathcal{Q}A) + \frac{g}{3} A \star A \star A \right). \quad (1)$$

Here there are three ingredients: an associative star product \star , the “trace” Tr , and the kinetic operator \mathcal{Q} . This action is similar to the Chern-Simons action. In this analogy $A(\bar{x})$ is similar to a 1-form which is a matrix and is a function of the string midpoint \bar{x} , \mathcal{Q} behaves like an exterior derivative, \star is like a matrix product, and Tr is like a matrix trace. Just like the Chern-Simons theory, the action is gauge invariant under

$$\delta A = \mathcal{Q}\Lambda + \Lambda \star A - A \star \Lambda. \quad (2)$$

The specific form of \mathcal{Q} defines the kinetic term, which corresponds to a description of a vacuum in the absence of the cubic interaction. If the cubic term is treated perturbatively, each choice of \mathcal{Q} defines a different perturbative expansion. The perturbative vacuum around any conformal theory background is given by the standard BRST operator constructed from ghosts and the Virasoro operators in that background (e.g. the flat background). However, one may expand the theory around any nonperturbative classical solution of the action, including solutions that correspond to D-brane backgrounds. In that case \mathcal{Q} contains information about the D-brane solution while satisfying the desired properties, namely that it acts like an exterior derivative. The vacuum string field theory (VSFT) proposal [6] corresponds to the closed string tachyon vacuum, and there is strong evidence that this is represented by a \mathcal{Q} that depends only the midpoint coordinates.

There are several realizations of $\star, \text{Tr}, \mathcal{Q}$ in various formalisms, including conformal field theory and oscillators [1], which have proven to be cumbersome to manipulate in computations, although they are perfectly correct. In MSFT the star \star is the standard Moyal product in half of the phase space of the string (even modes or related, see below) and the trace is the integral over the phase space. \mathcal{Q} maintains the same structure as other formalisms but is rewritten in terms of half the phase space, and becomes a second order differential operator in noncommutative space. With these ingredients string field theory becomes essentially field theory in noncommutative space, albeit with a kinetic term that is more involved than the usual noncommutative field theory. Thus MSFT is a much more familiar setting which is accessible to a

larger audience that has been studying noncommutative geometry, deformation or geometric quantization, and the associated field theories. Techniques employed in these other fields can now be borrowed to study string field theory. There are however, fascinating new properties not seen before in noncommutative field theory, including two timelike coordinates³ and the correct amount of gauge symmetry to remove related ghosts (negative norm states caused by the timelike coordinates).

In vacuum string field theory \mathcal{Q} may be as simple as $\mathcal{Q}\left(\frac{\pi}{2}\right) \sim c^+\left(\frac{\pi}{2}\right) + c^-\left(\frac{\pi}{2}\right)$, where $c^\pm(\sigma)$ are fermionic ghosts [6]. In the bosonized ghost approach³ this midpoint $\mathcal{Q}\left(\frac{\pi}{2}\right)$ becomes a simple differential operator [19]

$$\mathcal{Q}\left(\frac{\pi}{2}\right) \sim e^{i\bar{\phi}} \sin\left(\frac{\pi}{4} - \frac{\pi}{2} \frac{-i\partial}{\partial\bar{\phi}}\right). \quad (3)$$

It depends only on the midpoint of the bosonized ghost $\bar{\phi}$, while $\frac{-i\partial}{\partial\bar{\phi}}$ is the ghost number operator. For fields of definite ghost number we have⁴ $A(\bar{\phi}) \sim e^{i(n-\frac{1}{2})\bar{\phi}} A_{n-\frac{1}{2}}$, $n \in \mathbb{Z}$, where $A_{n-\frac{1}{2}}$ is independent of the midpoint $\bar{\phi}$. On this

³Traditional noncommutative field theory has only coordinates with Euclidean signature. However, in the MSFT approach we find that string field theory is a noncommutative field theory that contains 4 timelike coordinates for each even mode, namely x_e^μ, p_e^μ when $\mu = 0, 0'$. The label $\mu = 0$ corresponds to the usual timelike coordinate. Effectively there is a second timelike coordinate $\mu = 0'$ associated with the bosonization of the bc -ghosts, $c = e^{i\phi}, b = e^{-i\phi}$. The field $\phi(\sigma)$ behaves like a 27th dimension $\phi = x^{27} = ix_{0'}$ which is actually timelike $x_{0'}$. The reason is that b, c are hermitian and also the string field $A(\xi^{\tilde{\mu}}, \xi^{27})$ is real under the complex conjugation operation $A^*(\xi^{\tilde{\mu}}, -\xi^{27}) = A(\xi^{\tilde{\mu}}, \xi^{27})$, where $\tilde{\mu}$ represents the usual coordinates. These are possible only if $\xi_i^{27} = (\phi_e, p_{\phi_e})$ are antihermitian, or $\phi(\sigma) = ix_{0'}(\sigma)$ with the timelike $x_{0'}(\sigma)$ hermitian like the other string coordinates. In conformal field theory the operator products of ϕ have spacelike signature, therefore $x_{0'}$ has timelike signature. Thus, in string field theory, if the bosonized ghost is treated like the other coordinates under hermitian conjugation, the spacetime index on $\xi_i^\mu = (x_e^\mu, p_e^\mu)$ is labelled as $\mu = 0, 1, \dots, 25, 0'$, with two times. The (25,2) Lorentz symmetry of $\eta_{\mu\nu}$ is broken because the center of mass of the ghost coordinate ϕ_0 is compactified⁴, and also the kinetic operator \mathcal{Q} treats the ghosts $\phi(\sigma) = ix_{0'}(\sigma)$ differently than the other 26 dimensions. Nevertheless, the Moyal star among the higher modes (x_e^μ, p_e^μ) , which determines the interactions, respects the (25,2) symmetry. Thus the extra timelike coordinate is a fact of life in the theory even in the absence of a full (25,2) symmetry. This situation is reminiscent of Two-Time physics field theory [23] which has a gauge symmetry quite similar to the one in MSFT. It is my approach to Two-Time Physics field theory that led me to the concept of the Moyal product in string field theory.

⁴To produce the effect of the bc ghosts, the zero mode of ϕ is compactified on the periodic interval $0 \leq \phi_0 \leq 4\pi$ so that the ghost number $p_0^\phi = \frac{-i\partial}{\partial\phi_0} = \frac{-i\partial}{\partial\phi}$ has eigenvalues $p_0 \rightarrow n - 1/2$ with $n \in \mathbb{Z}$.

space it is easy to see that this \mathcal{Q} squares to zero $\mathcal{Q}^2 = 0$, and it has trivial cohomology - i.e. any field that satisfies $\mathcal{Q}A = 0$ ($n=\text{odd}$) is of the form $A \sim \mathcal{Q}A'$ ($n'=\text{even}$). Therefore in this vacuum there are no open strings since no physical solutions exist to the linearized equations of motion $\mathcal{Q}A = 0$. In this formalism, the physical string field in the action has ghost number $-1/2$ ($n = 0$), i.e. $A(\bar{\phi}) \sim e^{-\frac{i}{2}\bar{\phi}} A_{-\frac{1}{2}}$, while the star product is defined with an extra midpoint phase $\exp(-i\frac{3}{2}\bar{\phi})$. If one follows the dependence on the midpoint $\bar{\phi}$ it is easy to see that $\bar{\phi}$ can be completely removed from the theory and \mathcal{Q} is then replaced by a constant when applied on the physical field. The constant can be absorbed into renormalizations of $A_{-\frac{1}{2}}$ and g . Then the classical equation of motion becomes the projector equation $A_{-\frac{1}{2}} \star A_{-\frac{1}{2}} = A_{-\frac{1}{2}}$ which has many nonperturbative solutions that include D-branes. Solutions to the projector equation are easily obtained when \star is the Moyal product in MSFT, and a general explicit class that includes the sliver, butterfly etc. is given in [11]. The bosonized $\mathcal{Q}(\frac{\pi}{2})$ in Eq.(3) gives the simplest version of the VSFT proposal. But the VSFT proposal is still under investigation and it appears that the midpoint structure may receive some more clarification [18].

When Q corresponds to the perturbative vacuum, the action in the Siegel gauge simplifies to

$$S = \int d\bar{x}^\mu \text{Tr} \left(\frac{1}{2} A * ((L_0 - 1) A) - \frac{g}{3} A * A * A \right) \quad (4)$$

where L_0 is the Virasoro operator in any exact conformal theory background. In the flat background it is a sufficiently simple differential operator (see below) that is easily manageable in MSFT as shown in Feynman graph computations in [18]. In backgrounds that are closely related to flat backgrounds (such as tori, orbifolds, etc.) we expect that the formalism is also reasonably manageable. The form of Eq.(4) makes it evident that MSFT is a noncommutative field theory of the usual kind, but in a direct product space of Moyal planes⁵.

⁵Should one worry that in this formulation of string field theory there are an infinite number of timelike derivatives generated by the Moyal star? The first thing to note is that “time” t in string theory refers to the center of mass timelike coordinate x_0^0 , and that in MSFT ∂_t is rewritten as a derivative with respect to the *midpoint only* $\partial_t = \partial_{\bar{x}^0}$. With respect to the midpoint there are a maximum of two time derivatives in L_0 , as in usual field theory (see Eq.(13)). The timelike derivatives with respect to the higher modes (x_e^μ, p_e^μ) , with $\mu = 0, 0'$, in the Moyal product are not the “time” derivatives, so these do not pose a problem. Therefore, MSFT does not suffer from multiple “time” derivatives with respect to the actual “time” parameter. One should worry about negative norm states associated with timelike directions of the higher modes, but string theory is expected to be free of

3 Equivalent star products $\star_e, \star_o, \star_T, \star_t, \star_\kappa, \star_\sigma$

We turn to the star product. By now it has become clear that the MSFT star product in [8][11] can be rewritten in many equivalent bases. I will describe a few of these below.

Recall that the open string is parametrized in terms of string modes

$$x^\mu(\sigma) = x_0^\mu + \sqrt{2} \sum_{n \geq 1} x_n^\mu \cos(n\sigma). \quad (5)$$

It will be useful to separate the modes into even x_e^μ , $e = 2, 4, 6, \dots$, and odd x_o^μ , $o = 1, 3, 5, \dots$. The string field in position space is denoted by $\psi(x_0, x_e, x_o)$. Witten's star product corresponds to sawing two strings into a third one in position space - this is a complicated expression which has been difficult to manipulate in computations.

The essential steps to the Moyal formulation of string field theory were given in [8]. It starts with working in the Fourier basis $A(\bar{x}, x_e, p_o) \leftrightarrow \psi(x_0, x_e, x_o)$ reached by transforming the x_o variable to Fourier space using the kernel $\exp(-\frac{2i}{\theta} x_o p_o)$. The parameter θ absorbs units and can be mapped to 1, if desired, with an appropriate choice of units of p_o . We also rewrite the center of mass mode in terms of the midpoint $\bar{x} = x(\frac{\pi}{2})$ and x_e as

$$x_0 = \bar{x} + w_e x_e. \quad (6)$$

Starting with the basis (x_e, p_o) we will be interested in other bases (x_e, p_e) , (x_o, p_o) , $(\tilde{x}_e, \tilde{p}_o)$, $(x(\sigma), p(\sigma))$, $(x(\kappa), p(\kappa))$ that are transformed into each other. The matrices T_{eo}, R_{oe}, w_e, v_o , given by

$$T_{eo} = \frac{4o}{\pi(o^2 - e^2)} i^{o-e+1}, \quad R_{oe} = \frac{4e}{\pi o(o^2 - e^2)} i^{o-e+1}, \quad w_e = \sqrt{2} i^{-e+2}, \quad v_o = \frac{2\sqrt{2}}{\pi} \frac{i^{o-1}}{o}. \quad (7)$$

play a special role and appear in all computations of physical quantities. These emerge from the properties of the even and odd trigonometric functions in Eq.(5). It is useful to note that they satisfy the matrix relations (a bar means transpose)

$$R = (\kappa_o)^{-2} \bar{T} (\kappa_e)^2, \quad R = \bar{T} + v\bar{w}, \quad v = \bar{T}w, \quad w = \bar{R}v. \quad (8)$$

negative norm states thanks to the gauge symmetry of Eq.(2) combined with the on-shell BRST condition that is equivalent to the equations of motion (including interaction).

where κ_e, κ_o are diagonal matrices that represent the string oscillator frequencies. For an infinite number of modes the frequencies are given by $\kappa_e = e$, $\kappa_o = o$, but in a cutoff theory described below the diagonal entries (κ_e, κ_o) can be more general functions of (e, o) while Eqs.(8) continue to hold.

A cutoff is needed in all formulations of SFT to resolve associativity anomalies [10]⁶. The cutoff consists of working with a finite number of string modes $n = 1, 2, \dots, 2N$ that have oscillator frequencies κ_n , and introducing finite $N \times N$ matrices T_{eo}, R_{oe}, w_e, v_o as functions of the diagonal matrix $\kappa = \text{diag}(\kappa_e, \kappa_o)$. For $\kappa_n = n$ and $N = \infty$, the finite matrices T, R, w, v reduce to the expressions in Eq.(7). The finite matrices T, R, w, v are introduced as the solutions of Eq.(8) which are taken as the defining relations, since these are also satisfied by the infinite matrices. These equations were uniquely solved at finite N in terms of arbitrary κ_n [11][18]

$$T_{eo} = \frac{w_e v_o \kappa_o^2}{\kappa_e^2 - \kappa_o^2}, \quad R_{oe} = \frac{w_e v_o \kappa_e^2}{\kappa_e^2 - \kappa_o^2}, \quad (9)$$

$$w_e = i^{2-e} \frac{\prod_{o'} |\kappa_e^2 / \kappa_{o'}^2 - 1|^{\frac{1}{2}}}{\prod_{e' \neq e} |\kappa_e^2 / \kappa_{e'}^2 - 1|^{\frac{1}{2}}}, \quad v_o = i^{o-1} \frac{\prod_{e'} |1 - \kappa_o^2 / \kappa_{e'}^2|^{\frac{1}{2}}}{\prod_{o' \neq o} |1 - \kappa_o^2 / \kappa_{o'}^2|^{\frac{1}{2}}}. \quad (10)$$

Although the finite matrices are given quite explicitly, most computations are done by using simple matrix relations among T, R, w, v without the need for their explicit form. The following matrix relations are derived from Eq.(8) and therefore they apply at finite as well as infinite N for all choices of κ_n as a function of n

$$\begin{aligned} TR &= 1_e, \quad RT = 1_o, \quad \bar{R}R = 1_e + w\bar{w}, \quad \bar{T}T = 1_o - v\bar{v}, \\ T\bar{T} &= 1_e - \frac{w\bar{w}}{1 + \bar{w}w}, \quad Tv = \frac{w}{1 + \bar{w}w}, \quad \bar{v}v = \frac{\bar{w}w}{1 + \bar{w}w}, \\ R\bar{w} &= v(1 + \bar{w}w), \quad R\bar{R} = 1_o + v\bar{v}(1 + \bar{w}w). \end{aligned} \quad (11)$$

It is important to emphasize that in this formalism computing with arbitrary frequencies κ_n and finite number of modes $2N$, is as easy as working directly in the limit. Thus, in the following the finite matrices with arbitrary κ_n, N will be assumed unless otherwise specified.

⁶As seen from Eq.(7) for large N we get $\bar{w}w \rightarrow 2N \rightarrow \infty$. This behavior causes associativity anomalies in multiple matrix products. As an illustration consider the matrix product RTv : if we use $RT = 1$ and $Tv = 0$ which are valid at large N , we get $(RT)v = 1 \times v = v$, versus $R(Tv) = R \times 0 = 0$. Such associativity anomalies are avoided by computing all matrix products at finite N , and taking the limit only at the end. Note from Eq.(11) that the factor $(1 + \bar{w}w) \rightarrow \infty$ appears in the denominator of Tv but in the numerator of Rw . Such factors cancel each other in multiple products and resolve the associativity anomalies.

Even base \star_e : In [8] it was shown that it is possible to disentangle Witten's star into independent Moyal stars for each mode e by defining p_e through the equation $p_o = p_e T_{eo}$. Then the string field $A(\bar{x}, x_e, p_o)$ is rewritten in the even phase space as $A(\bar{x}, x_e, p_e)$ and the Moyal star is diagonalized $A \star_e B$ in the noncommutative space $\xi_i^\mu = (x_e^\mu, p_e^\mu)$

$$\star_e = e^{\eta_{\mu\nu} \sigma^{ij} \overleftarrow{\partial}_i^\mu \overrightarrow{\partial}_j^\nu}, \quad \sigma_{ij} = i\theta \begin{pmatrix} 0 & 1_e \\ -1_e & 0 \end{pmatrix}. \quad (12)$$

While the Moyal star is diagonal in this basis, the Virasoro operator L_0 , which determines the perturbative string spectrum, is not. It is given by [11][18]

$$L_0 = \frac{1}{2}\beta_0^2 - \frac{d}{2}Tr(\tilde{\kappa}) - \frac{1}{4}\bar{D}_\xi(M_0^{-1}\tilde{\kappa})D_\xi + \bar{\xi}(\tilde{\kappa}M_0)\xi, \quad (13)$$

where $\beta_0 = -il_s \frac{\partial}{\partial \bar{x}}$, $D_\xi = \left(\left(\frac{\partial}{\partial x_e} - i \frac{\beta_0}{l_s} w_e \right), \frac{\partial}{\partial p_e} \right)$, and $l_s = \sqrt{2\alpha'}$ is the string length. The matrices

$$\tilde{\kappa} = \begin{pmatrix} \kappa_e & 0 \\ 0 & T\kappa_o R \end{pmatrix}, \quad M_0 = \begin{pmatrix} \frac{\kappa_e}{2l_s^2} & 0 \\ 0 & \frac{2l_s^2}{\theta^2} T\kappa_o^{-1} \bar{T} \end{pmatrix}, \quad (14)$$

give the block diagonal forms

$$M_0^{-1}\tilde{\kappa} = \begin{pmatrix} 2l_s^2 1_e & 0 \\ 0 & \frac{\theta^2}{2l_s^2} \kappa_e^2 \end{pmatrix}, \quad \tilde{\kappa}M_0 = \begin{pmatrix} \frac{1}{2l_s^2} \kappa_e^2 & 0 \\ 0 & \frac{2l_s^2}{\theta^2} T\bar{T} \end{pmatrix} \quad (15)$$

after using Eqs.(8,11). Every term in L_0 has diagonal matrices except for the last term. Note that $T\bar{T}$ in $\tilde{\kappa}M_0$ is almost diagonal, since $T\bar{T} = 1 - \frac{w\bar{w}}{1+\bar{w}w}$, and the second term becomes naively negligible in the large N limit since $\bar{w}w \rightarrow \infty$. A major simplification would occur if one could neglect this term as $N \rightarrow \infty$. However, due to the anomalies⁶ neglecting this term gives wrong results⁷. The lesson is that the large N limit should not be taken in the Lagrangian and should wait until the end of computations, as seen in explicit examples in [18].

Most of the computations in MSFT were carried out [8][10][11][18] in this basis, but they may also be carried out with the same ease in any of the equivalent bases discussed in this section.

⁷The extra term in L_0 proportional to $(1 + \bar{w}w)^{-1}$ was missed in [12] in their attempt to compare the discrete Moyal \star_e of [8] to the continuous Moyal \star_κ directly at $N = \infty$, and erroneously concluded that there was a discrepancy in the string spectrum. In fact, this term is not negligible, it contributes through the anomaly described in footnote (6) to produce the correct spectrum.

Odd base \star_o : In [8] the odd phase space, (x_o^μ, p_o^μ) was also apparent. One may introduce x_o through $x_e = T_{eo}x_o$ and write the same string field $A(\bar{x}, x_e, p_o)$ in terms of the odd phase space $A(\bar{x}, x_o, p_o)$. Witten's star is again disentangled into independent Moyal stars $A \star_o B$ for each o in the non-commutative space $\xi_I = (x_o, p_o)$

$$\star_o = e^{\eta_{\mu\nu}\sigma^{ij}\overleftarrow{\partial}_I^\mu\overrightarrow{\partial}_J^\nu}, \quad \sigma_{IJ} = i\theta \begin{pmatrix} 0 & 1_o \\ -1_o & 0 \end{pmatrix} \quad (16)$$

The relation between the odd/even variables

$$p_o = p_e T_{eo}, \quad x_o = R_{oe} x_e \quad (17)$$

is a canonical transformation that leaves the Moyal product invariant $\star_e = \star_o$. Furthermore, using Eqs.(8,17) we can write $\bar{x} = x_o + v_o x_o$. This substitution must be made before computing the star \star_o . Note in particular that $[x_o, p_{o'}]_{\star_e} = [x_o, p_{o'}]_{\star_o} = i\theta\delta_{oo'}$. Thus, (x_o^μ, p_o^μ) are regarded as functions of (x_e^μ, p_e^μ) , so that the even/odd phase spaces do not star commute with each other.

The Virasoro operator L_0 is easily rewritten in the new basis by applying the canonical transformation in Eq.(17). The result is similar to Eq.(13,15), with the replacement of the odd basis, with the transformed $M_0^{-1}\tilde{\kappa}, \tilde{\kappa}M_0$

$$M_0^{-1}\tilde{\kappa} \rightarrow \begin{pmatrix} 2l_s^2 (R\bar{R})_{oo'} & 0 \\ 0 & \frac{\theta^2}{2l_s^2} \kappa_o^2 \end{pmatrix}, \quad \tilde{\kappa}M_0 \rightarrow \begin{pmatrix} \frac{\kappa_o^2}{2l_s^2} & 0 \\ 0 & \frac{2l_s^2}{\theta^2} 1_o \end{pmatrix}, \quad (18)$$

where we used $(\kappa_o)^2 = \bar{T}(\kappa_e)^2 T$ which follows from Eqs.(8)). Note that $R\bar{R} = 1_o + v\bar{v}(1 + \bar{w}w)$ is almost diagonal, but has the diverging term with $(1 + \bar{w}w)$ in the numerator. By contrast, in the even case this factor appeared in $T\bar{T}$ in the denominator. Again, this divergence does not actually occur in computations because it gets killed by a zero. By waiting to take the limit until the end of the computation such anomalous terms are correctly treated, and give the correct spectrum of L_0 . One may choose either \star_e or \star_o to formulate MSFT at finite N , and obtain the same results at finite or infinite N .

Mixed bases \star_T, \star_t : One may also define MSFT in the original mixed even/odd basis $A(\bar{x}, x_e, p_o)$. Then, the Virasoro operator L_0 is diagonal, but the Moyal star is not diagonal [8]. It is given by

$$A \star_T B = A(\bar{x}, x_e, p_o) \exp \left\{ i\theta \eta_{\mu\nu} T_{eo} \left(\overleftarrow{\partial}_{x_e}^\mu \overrightarrow{\partial}_{p_o}^\nu - \overleftarrow{\partial}_{p_o}^\nu \overrightarrow{\partial}_{x_e}^\mu \right) \right\} B(\bar{x}, x_e, p_o), \quad (19)$$

which gives $[x_e, p_o]_{\star_e \text{ or } \star_o} = [x_e, p_o]_{\star_T} = i\theta T_{eo}$. In this basis the noncommutative product $\star_T = \exp\left(\sigma_T \overleftarrow{\partial} \cdot \overrightarrow{\partial}\right)$ involves the antisymmetric matrix

$$\sigma_T = i\theta \begin{pmatrix} 0 & T \\ -\bar{T} & 0 \end{pmatrix} \quad (20)$$

In this basis L_0 is diagonal and takes the form of Eq.(13) with the replacement $\xi \rightarrow (x_e, p_o)$ and the matrices in Eq.(15) are mapped to diagonal matrices

$$M_0^{-1} \tilde{\kappa} \rightarrow \begin{pmatrix} 2l_s^2 1_e & 0 \\ 0 & \frac{\theta^2}{2l_s^2} \kappa_o^2 \end{pmatrix}, \quad \tilde{\kappa} M_0 \rightarrow \begin{pmatrix} \frac{\kappa_e^2}{2l_s^2} & 0 \\ 0 & \frac{2l_s^2}{\theta^2} 1_o \end{pmatrix}. \quad (21)$$

The dependence on the frequencies κ_e, κ_o can be partially shifted to the star product by a rescaling of the mixed phase space

$$\tilde{x}_e = \kappa_e^{1/2} x_e, \quad \tilde{p}_o = p_o \kappa_o^{-1/2}, \quad (22)$$

A new star product \star_t can be defined by using the rescaled T

$$t_{eo} = \kappa_e^{1/2} T_{eo} \kappa_o^{-1/2} = \frac{\kappa_e^{1/2} w_e v_o \kappa_o^{3/2}}{\kappa_e^2 - \kappa_o^2} \quad (23)$$

as

$$A \star_t B = A(\bar{x}, \tilde{x}_e, \tilde{p}_o) \exp\left\{i\theta \eta_{\mu\nu} t_{eo} \left(\overleftarrow{\partial}_{\tilde{x}_e}^\mu \overrightarrow{\partial}_{\tilde{p}_o}^\nu - \overleftarrow{\partial}_{\tilde{p}_o}^\nu \overrightarrow{\partial}_{\tilde{x}_e}^\mu\right)\right\} B(\bar{x}, \tilde{x}_e, \tilde{p}_o) \quad (24)$$

The rescaled phase space satisfies $[\tilde{x}_e, \tilde{p}_o]_{\star_e \text{ or } \star_o} = [\tilde{x}_e, \tilde{p}_o]_{\star_t} = i\theta t_{eo}$, and in the new basis the noncommutative product $\star_t = \exp\left(\sigma_t \overleftarrow{\partial} \cdot \overrightarrow{\partial}\right)$ involves the antisymmetric matrix

$$\sigma_t = i\theta \begin{pmatrix} 0 & t \\ -\bar{t} & 0 \end{pmatrix} \quad (25)$$

In this basis L_0 remains diagonal and takes the form of Eq.(13) with the replacement $\xi \rightarrow (\tilde{x}_e, \tilde{p}_o)$ and

$$M_0^{-1} \tilde{\kappa} \rightarrow \begin{pmatrix} \frac{2l_s^2}{\kappa_e} & 0 \\ 0 & \frac{\theta^2 \kappa_o}{2l_s^2} \end{pmatrix}, \quad \tilde{\kappa} M_0 \rightarrow \begin{pmatrix} \frac{\kappa_e}{2l_s^2} & 0 \\ 0 & \frac{2l_s^2}{\theta^2 \kappa_o} \end{pmatrix}. \quad (26)$$

κ base \star_κ : Motivated by the spectroscopy of Neumann coefficients in SFT, and guided by the Moyal product \star_e in [8], a continuous Moyal basis $(x(\kappa), p(\kappa))$ was suggested in [12]. This was defined directly for $N \rightarrow \infty$ and it was evident that in explicit computations a regulator would be needed at $\kappa = 0$ to resolve issues such as an unpaired zero mode ($x(0) = 0, p(0) \neq 0$) that is closely related to the anomalous zero mode which was discussed earlier in [10]. A convenient regulator is the one already suggested before in [10]. Indeed, it was shown in [11], that the continuous κ -basis is reached from a regulated discrete k -basis by diagonalizing the finite $N \times N$ matrix t that appears in \star_t

$$t_{eo} = \sum_{k=1}^N (V^{(e)})_{ek} \tau_k (\bar{V}^{(o)})_{ko} \rightarrow \int_0^\infty d\kappa V_e(\kappa) \left(\tanh \frac{\pi\kappa}{4} \right) V_o(\kappa), \quad (27)$$

where the matrices $V^{(e)}, V^{(o)}$ are orthogonal and τ_k are the eigenvalues (all functions of the arbitrary κ_n, N). At $N = \infty$ and $\kappa_n = n$ the discrete k basis becomes the continuous κ basis, the eigenvalues become $\tau_k \rightarrow \tanh \frac{\pi\kappa}{4}$ and the matrix elements of $V^{(e)}, V^{(o)}$ become continuous functions of κ , as given in [12][11].

The phase space in the discrete k basis (x_k, p_k) is related to the previous discrete bases by

$$x_k = \tilde{x}_e (V^{(e)})_{ek} = x_e \kappa_e^{1/2} (V^{(e)})_{ek}, \quad (28)$$

$$p_k = \frac{2}{\theta} \tilde{p}_o (V^{(o)})_{ok} = \frac{2}{\theta} p_o \kappa_o^{-1/2} (V^{(o)})_{ok} = \frac{2}{\theta} p_e \kappa_e^{-1/2} (V^{(e)})_{ek} \tau_k \quad (29)$$

The previous star products give

$$[x_k, p_{k'}]_{\star_t \text{ or } \star_e} = 2\tau_k \delta_{kk'} \equiv [x_k, p_{k'}]_{\star_\kappa} \quad (30)$$

where the right hand side defines $\star_\kappa = \exp \left(\sigma_\kappa \overleftarrow{\partial} \cdot \overrightarrow{\partial} \right)$ with

$$\sigma_\kappa = i \begin{pmatrix} 0 & 2\tau_k \\ -2\tau_k & 0 \end{pmatrix} \quad (31)$$

In this basis L_0 is not diagonal and takes the form of Eq.(13) with the replacement $\xi \rightarrow (x_k, p_k)$ and

$$M_0^{-1} \tilde{\kappa} \rightarrow \begin{pmatrix} \bar{V}^{(e)} \frac{2l_s^2}{\kappa_e} V^{(e)} & 0 \\ 0 & V^{(o)} \frac{2\kappa_o}{l_s^2} \bar{V}^{(o)} \end{pmatrix}, \quad \tilde{\kappa} M_0 \rightarrow \begin{pmatrix} \bar{V}^{(e)} \frac{\kappa_e}{2l_s^2} V^{(e)} & 0 \\ 0 & V^{(o)} \frac{l_s^2}{2\kappa_o} \bar{V}^{(o)} \end{pmatrix}. \quad (32)$$

At $N = \infty$ this discrete star product becomes the continuous star product of [12]. The continuous version of Eq.(30) is

$$[x(\kappa), p(\kappa')]_{\star_\kappa} = 2 \tanh \frac{\pi\kappa}{4} \delta(\kappa - \kappa'). \quad (33)$$

This shows an anomalous behavior at $\kappa = 0$ since the right hand side vanishes $\tanh \frac{\pi\kappa}{4} \rightarrow 0$. A study of the functions $V_e(\kappa), V_o(\kappa)$ show that for small κ we get $x(\kappa) \sim \kappa \rightarrow 0$, but $p(\kappa) \rightarrow p(0)$ tends to the zeroth power of κ , indicating the presence of an anomalous zero mode. More precisely, by noting that $V_o(0) = \frac{\sqrt{\pi o}}{4} v_o$ where v_o is given in Eq.(7), and using Eq.(29) in the $N \rightarrow \infty$ limit, we obtain the relation between the anomalous mode in the continuous basis and the one discussed in [10] which was $\tilde{p}^\mu = \sum_{o>0} v_o p_o^\mu$, thus

$$\frac{2\theta}{\sqrt{\pi}} p^\mu(0) = \sum_{o>0} v_o p_o^\mu = \sum_{o,e,e'>0} \bar{w}_e T_{eo} \bar{T}_{oe'} p_{e'}^\mu = \frac{1}{1 + \bar{w}w} \sum_{e>0} w_e p_e^\mu = \tilde{p}^\mu. \quad (34)$$

As we emphasized several times before it is dangerous to set $(1 + \bar{w}w) \rightarrow \infty$ before completing a computation, so $\frac{2\theta}{\sqrt{\pi}} p(0) = \tilde{p}$ does not necessarily vanish. In fact, it is related to the structure $\frac{1}{1 + \bar{w}w} (\sum_{e>0} w_e p_e)^2$ which is the anomalous part of L_0 that emerges from $T\bar{T} = 1 - \frac{w\bar{w}}{1 + \bar{w}w}$ in Eq.(15). This piece in L_0 plays a nontrivial role in producing the correct perturbative string spectrum and the correct propagator [11][18]. Thus $p(0)$ is precisely the anomalous mode \tilde{p} which needs to be treated just as carefully in computations in any basis.

In particular, note that the center of mass coordinate $x_0 = \bar{x} + w_e x_e$ does not commute with the anomalous momentum mode \tilde{p} since

$$[x_0, \tilde{p}]_{\star_e} = \sum_{o,e>0} [w_e x_e, v_o p_o]_{\star_e} = i\theta \sum_{o,e>0} w_e T_{eo} v_o = i\theta \bar{v}v = i\theta \frac{\bar{w}w}{1 + \bar{w}w} \rightarrow i\theta. \quad (35)$$

If the $N \rightarrow \infty$ limit is taken at intermediate steps (as in footnote (6)), ambiguous results follow depending on whether the e or the o sum is done first - doing the o sum first would give 0 instead of Eq.(35). So if one is working directly in the $N = \infty$ limit, should one give the priority to defining the center of mass x_0 (e first) or the anomalous mode \tilde{p} (o first)? Evidently both are important and the question needs a careful answer. We see that $p(0)$ is clearly anomalous and not a constant after all, since when carefully handled it does translate the center of mass (and has other similar anomalous effects). As such, it participates in gauge transformations that are related to

general coordinate transformations, an indication that closed strings relate to the anomalous mode $p(0) \sim \tilde{p}$, as emphasized in [10].

All this shows that one must be careful when computing in the continuous basis. The regulated discrete version of the continuous basis given in Eq.(30) and in [11] provides a careful approach. It is evident that with proper care the continuous Moyal product \star_κ is fully equivalent to the other discrete Moyal products $\star_e, \star_o, \star_t$.

Sigma base \star_σ : The σ -base is directly related to the worldsheet parameter σ . The same string field $A(\bar{x}, x_e, p_o)$ or $A(\bar{x}, x_o, p_o)$ is expressed in continuous phase space $A(\bar{x}, x(\sigma), p(\sigma))$ by using the orthogonal transformation

$$x(\sigma) = \sqrt{2} \sum_{o>0} x_o \cos \kappa_o \sigma, \quad p(\sigma) = \frac{2\sqrt{2}}{\pi} \sum_{o>0} p_o \cos \kappa_o \sigma, \quad 0 \leq \sigma \leq \frac{\pi}{2}, \quad (36)$$

For $N = \infty$ and $\kappa_o = o$, the odd cosines $\cos o\sigma$ form a complete set of functions in the range $0 \leq \sigma \leq \frac{\pi}{2}$ with Neumann boundary conditions at $\sigma = 0$ and Dirichlet boundary conditions at $\sigma = \frac{\pi}{2}$. Note that, since σ stops at the midpoint, the phase space $(x(\sigma), p(\sigma))$ is half of the full string phase space, and excludes the midpoint. Therefore under either the even \star_e or odd \star_o they produce a continuous delta function

$$[x(\sigma), p(\sigma')]_{\star_e \text{ or } \star_o} = [x(\sigma), p(\sigma')]_{\star_\sigma} = i\theta \delta(\sigma - \sigma') \quad (37)$$

$$\delta(\sigma - \sigma') = \frac{4}{\pi} \sum_{o>0} \cos o\sigma \cos o\sigma', \quad 0 \leq \sigma \leq \frac{\pi}{2} \quad (38)$$

Therefore, Eq.(36) is a canonical transformation, and allows us to define the continuous σ Moyal basis with \star_σ defined by the right hand side of Eq.(37).

A discrete version of the sigma basis can also be defined by taking N discrete points σ_k , $k = 1, \dots, N$ in the interval $0 \leq \sigma_k \leq \frac{\pi}{2}$ and correspondingly choosing κ_e, κ_o so that $\cos(\kappa_o \sigma_k)$ is an orthogonal transformation acting on an N dimensional basis. A discrete basis with such properties can be constructed with the methods of [24].

The σ basis may be convenient to discuss the tensionless string limit [15]⁸. We note that in the tensionless limit the spectrum of strings produce an infinite

⁸In an attempt to apply string field theory to the pp-wave background in a lightcone formalism [15] the continuous \star_σ basis was arrived at with a series of tentative arguments that are less transparent than the straightforward transformation given in Eq.(36).

number of massless high spin fields ⁹. Formulating an interacting theory of massless high spin fields has been a challenge. However, by doing perturbation theory for a small tension (described by large $l_s = \sqrt{2\alpha'}$ expansion of the kinetic term L_0) we seem obtain a theory of interacting massless high spin fields, which should be physically consistent, since it is a different expansion of a consistent string theory.

4 Summary of computations

In [11] it is shown that for most computations of interest one should consider the field configurations that contain the general parameters $\mathcal{N}, M_{ij}, \lambda_i^\mu, k^\mu$

$$A_{\mathcal{N}, M, \lambda, k}(\bar{x}, \xi) = \mathcal{N} \exp(-\bar{\xi} M \xi - \bar{\xi} \lambda + i k \cdot \bar{x}), \quad (39)$$

For perturbative states M is replaced by M_0 of Eq.(14), but for D-brane states $(M\sigma)^2 = 1$. The $A_{\mathcal{N}, M, \lambda, k}(\bar{x}, \xi)$ are the only field configurations that are needed to compute Feynman graphs with perturbative or nonperturbative external states [18]. These fields form a closed algebra under the star

$$\begin{aligned} & (\mathcal{N}_1 \exp(-\bar{\xi} M \xi - \bar{\xi} \lambda + i k_1 \cdot \bar{x})) \star (\mathcal{N}_2 \exp(-\bar{\xi} M_2 \xi - \bar{\xi} \lambda_2 + i k_2 \cdot \bar{x})) \\ &= (\mathcal{N}_{12} \exp(-\bar{\xi} M_{12} \xi - \bar{\xi} \lambda_{12} + i (k_1 + k_2) \cdot \bar{x})) \end{aligned} \quad (40)$$

where the structure of $\mathcal{N}_{12}, (M_{12})_{ij}, (\lambda_{12})_i^\mu$ is given as [8][11] (define $m_1 = M_1 \sigma, m_2 = M_2 \sigma, m_{12} = M_{12} \sigma$)

$$m_{12} = (m_1 + m_2 m_1) (1 + m_2 m_1)^{-1} + (m_2 - m_1 m_2) (1 + m_1 m_2)^{-1}, \quad (41)$$

$$\lambda_{12} = (1 - m_1) (1 + m_2 m_1)^{-1} \lambda_2 + (1 + m_2) (1 + m_1 m_2)^{-1} \lambda_1, \quad (42)$$

$$\mathcal{N}_{12} = \frac{\mathcal{N}_1 \mathcal{N}_2}{\det(1 + m_2 m_1)^{d/2}} e^{\frac{1}{4}((\bar{\lambda}_1 + \bar{\lambda}_2)\sigma(m_1 + m_2)^{-1}(\lambda_1 + \lambda_2) - \bar{\lambda}_{12}\sigma(m_{12})^{-1}\lambda_{12})}. \quad (43)$$

The algebra (40) is a monoid, which means it is associative, closed, and includes the identity element ($A_{1,0,0,0} = 1$). It is short of being a group since some elements (in particular projectors) do not have an inverse, although the generic element does have an inverse. The trace of a monoid is given by

$$Tr(A_{\mathcal{N}, M, \lambda, k}) = \frac{\mathcal{N} e^{i k \cdot \bar{x}} e^{\frac{1}{4} \bar{\lambda} M^{-1} \lambda}}{\det(2M\sigma)^{d/2}}. \quad (44)$$

⁹Massless high spin fields have been discussed in [25] using a Moyal star product. The Moyal basis in [25] may be thought of as the twistor version of phase space (x^μ, p^μ) , hence there is a close relation. In string theory there are many copies of the phase space, one for each e , and therefore many more high spin fields. Furthermore, string fields are functions of the midpoint degree of freedom \bar{x} .

This monoid structure was used as a computational tool in [11] to calculate explicitly the following quantities (with perturbative M_0 , or any nonperturbative M)

- Static field configurations of interest, including: “Wedge” fields that correspond to powers of a single monoid $W_n(\xi) = (A_{\mathcal{N},M,\lambda,0})_\star^n$; Sliver field that corresponds to the infinite power of the perturbative vacuum $\Xi(\xi) = (A_{\mathcal{N}_0,M_0,0,0})_\star^\infty$; D-brane vacua described by projectors that satisfy $A \star A = A$, with $\text{Tr} A = 1$. These projectors take the general form $A_{D,\lambda}(\xi) = \mathcal{N} \exp(-\bar{\xi} D \xi - \bar{\xi} \lambda)$ with

$$\mathcal{N} = \left(\prod_{e>0} 2^d \right) \exp\left(-\frac{1}{4} \bar{\lambda} \sigma D \sigma \lambda\right), \quad D = \begin{pmatrix} a & ab \\ ba & \frac{1}{\theta^2 a} + bab \end{pmatrix}, \quad (45)$$

for any λ and any symmetric a, b . The components of λ^μ parallel to the D-brane vanish $\lambda^\parallel = 0$, while those perpendicular to the D-brane are functions of the transverse components of the midpoint $\lambda^\perp(\bar{x}_\perp) \neq 0$. The “Sliver”, “Butterfly” etc. are special cases of the above form.

- All n -point interaction vertices $\text{Tr}(A_1 \star \cdots \star A_n)$ in the cutoff theory were computed for perturbative or nonperturbative monoids, for any frequencies κ_e, κ_o, N . The simplicity of such computations is one of the payoffs of the reformulation provided by MSFT. For $N \rightarrow \infty$ these computations reproduced and generalized many results that were obtained through other methods and produced new ones that were computed for the first time. Such explicit analytic results, especially at finite N , are new, and not obtained consistently in any other approach. At finite N the MSFT results could be used in numerical as well as analytic computations as a more consistent method than level truncation.
- As a test of MSFT, Neumann coefficients used in the oscillator formulation for any number of strings were computed as a corollary of the n -point vertices mentioned above. Previous computations of these coefficients relied on conformal field theory, which yielded expressions that were difficult to manipulate, and only for a few strings. The MSFT computation was done with arbitrary oscillator frequencies κ_n and cutoff N . The cutoff version of Neumann coefficients $N_{mn}^{rs}(t), N_{0n}^{rs}(t, w), N_{00}^{rs}(t, w)$, were found to be simple analytic expressions that depend on the single $N \times N$ matrix $t_{eo} = \kappa_e^{1/2} T_{eo} \kappa_o^{-1/2}$ and the N -vector w_e . These explicitly satisfy the Gross-Jevicki nonlinear relations for any κ_n, N . It is then evident

that T and w are more fundamental than the Neumann coefficients. As a corollary of this result, by diagonalizing the matrix t as in Eq.(8) one can easily understand at once why there is a Neumann spectroscopy for the 3-point vertex [26] or more generally the n -point vertex [11]. The MSFT result for n strings agreed with the conformal field theory computation whenever the latter were available.

- The oscillator and Virasoro algebra were constructed from fields under the star product. The fields $\mathcal{L}_n(\xi)$ were constructed from half the phase space ξ (unlike the usual construction that uses the full oscillator space). These are fields, not differential operators, and they close under the star commutator to form the Virasoro algebra in noncommutative space. The exponentiated Virasoro fields form a subset of the monoid algebra that corresponds to a group.
- Tachyon condensation and small fluctuations in VSFT were investigated (for an independent approach see [16]). In [11] it was concluded that at finite N the tachyon equation $\Xi \star T + T \star \Xi = T$ had only pure gauge solutions, and that at $N = \infty$ the associativity anomaly needed to play a role for the VSFT program to succeed. In later work [27] it is found that it is possible to construct the perturbative tachyon from VSFT provided the tachyon is guessed as a configuration that almost obeys the VSFT equations as $N \rightarrow \infty$ thanks to the anomaly. In the notation of [11] the tachyon fluctuation around the sliver field $\Xi(\xi)$ is given by $T(\bar{x}, \xi) = \frac{2^{-3/2}}{\sqrt{2\beta K}} \int d^d k \, n(k) e^{-\frac{1}{4}\bar{\lambda}(k)\sigma m \lambda(k)} \Xi(\xi) e^{-\xi \lambda(k)} e^{ik \cdot \bar{x}}$, where $n(k)$ is the normalized tachyon wavefunction in momentum space, while

$$\lambda^\mu(k) = k^\mu \begin{pmatrix} (a^{1/2}w)_e \\ 0 \end{pmatrix} \sqrt{\frac{8}{\bar{w}w} \frac{\ln(F(k^2)/2)}{k^2}}, \quad (46)$$

where a is the matrix in the sliver field (as in Eq.(45) with $b = 0$). The tachyon equation is obeyed since $\lambda^\mu(k) \rightarrow 0$ as $\bar{w}w \rightarrow 2N \rightarrow \infty$. But note that $\exp(\frac{1}{8}\bar{\lambda}\sigma m \lambda) = \frac{1}{2}F(k^2)$ is finite. This is the anomaly. By taking a form factor whose behavior near the tachyon mass shell $k^2 \sim -1$ is

$$F(k^2) \approx 1 + \beta(k^2 + 1) + O((k^2 + 1)^2), \text{ with } \beta = \frac{1}{2} \left(\frac{\pi^2}{3} \right)^{1/3}, \quad (47)$$

we find the correct relation between the D25-brane tension and the

tachyon coupling

$$g_T = \frac{1}{\sqrt{\tilde{K}}} \frac{1}{(2\beta)^{3/2}}, \quad T_{25} = \frac{\tilde{K}}{6}, \quad \rightarrow \quad T_{25} = \frac{1}{2\pi^2} \frac{1}{g_T^2}. \quad (48)$$

What remains to be understood in this problem is the properties of the form factor $F(k^2)$ given above. But note that the midpoint structure of VSFT may receive some correction [18], and this may also shed some light on the remaining issues in this problem.

- More recently the framework for computations of Feynman graphs using MSFT was presented in [18]². Efficient analytic methods of computation were developed based on (a) the monoid algebra in noncommutative space and (b) the conventional Feynman rules in Fourier space. The methods apply equally well to perturbative string states or nonperturbative string states involving D-branes.
- The ghost sector was mostly handled in the bosonized version. However it can also be formulated using Moyal products with fermionic b, c ghosts [14][18]. The version in [18] takes into account the subtleties of the midpoint.

5 Challenges

Some open problems and challenges beyond the topics discussed above are the following:

- Construction of BRST operator in MSFT with fermionic or bosonized ghosts. This is straightforward for $N = \infty$, but a version that applies at finite N in the cutoff theory is still lacking because a substitute for the Virasoro algebra that closes at finite N remains to be found. Using such a BRST operator nonperturbative solutions of the equations of motion that connect the perturbative vacuum to the nonperturbative (D-brane) vacua can be investigated more reliably in the cutoff theory.
- Closed string field configurations and their relation to associativity anomalies remain to be investigated. This may be understood eventually through the Feynman graph techniques given in [18].

- Formulation of supersymmetric MSFT. Our proposal is that this may be achieved by figuring out the Moyal-Weyl quantization of the Brink-Schwarz superparticle. Then MSFT fields would be taken as functions of $(x_e^\mu, p_e^\mu, \theta_e^\mu)$ with a supersymmetric Moyal product among them.
- Formulation of MSFT for strings propagating on curved backgrounds. Here we expect that the Kontsevich star product [28] plays a role. As we have seen in the discussion of various Moyal bases, the complications may be shifted from the star product to the propagator or vice versa. Naively, in curved backgrounds, it appears that all the burden may be put on a complicated L_0 while the star product could be chosen as the simple Moyal product based on the canonical structure of string theory (perhaps this ignores global properties of the curved space). However, an alternative to this could be to shift the burden to a complicated Kontsevich star product while maintaining a simpler form of L_0 . The relation between the geometric properties of the background that determines L_0 and the Kontsevich star product is currently poorly understood.
- A study of high spin massless fields by expanding MSFT around tensionless strings appears to be promising.
- SFT or MSFT look like gauge fixed from something else since they contain ghosts. What is the higher gauge symmetry and what is the higher symmetric form of MSFT before gauge fixing? We suspect that general canonical transformations in ξ space may provide an answer.
- Connection to 2T-physics. In footnote (3) it was already explained that MSFT with bosonized ghosts has a noncommutative space with $(25, 2)$ signature. There are some similarities between MSFT and 2T-physics field theory [23] and a deeper relation between them is suspected. The 2T approach displays higher hidden symmetries and provides a holographic view of the higher dimensional theory in the form of many dual lower dimensional theories. As recently demonstrated [29] $\text{AdS}_5 \times \text{S}^5$ supergravity can be viewed as a holographic picture of a 12-dimensional 2T theory with $(10, 2)$ signature. A lot more structure of this type in M-theory may be expected, and the MSFT and 2T techniques seem to be natural tools to discover it and formulate it.

MSFT demystifies string field theory by providing a more familiar structure in the form of noncommutative geometry with more efficient computational tools. There still are technical and conceptual challenges as outlined in this

lecture, but there is good reason to believe that the tools to make progress in those areas are available or can be developed. So we hope that string field theory will become a viable framework for defining the nonperturbative theory as well as for performing practical computations to relate it to low energy physics and cosmology.

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